SET CHAINS MOD THE PI-1-N ENFORCEABLE FILTER

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Abstract: In previous papers the author has considered set chains mod the club filter. Here they are considered mod the Π^1_n-enforceable filter. Remarks are made on repeat points, including sufficient conditions for their non-existence.

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1. Introduction

In a series of papers [2], [3], [4], [5], the author has considered set chains mod the club filter. Applications include new axioms for set theory, lower bounds on the Mahlo rank of a weakly compact cardinal, and lower bounds on repeat points.

Here set chains are considered mod the Π^1_n-enforceable filter. This further develops Hellsten's theory of generalized Mahlo operations, and provides new lower bounds on repeat points and other ordinals related to measurable cardinals.

Let Card denote the cardinals, and Inac the (strongly) inaccessible cardinals. Let $L_\in$ denote the language of set theory, and $L^s_\in$ the two-sorted language with second-order variables. A set may be considered as a structure for $L^s_\in$, and will be called an $\in$-structure.

Let Δ^1_∞ denote the formulas of $L^s_\in$ with all free variables second order; given $\kappa \in \mathrm{Card}$ second order parameters $P \subseteq V_\kappa$ may be allowed.

A function will be said to be Σ_1^{ZFC} if it is defined by a Σ_1 formula of $L_\in$, which provably in ZFC defines a function.

For a measurable cardinal κ and a normal ultrafilter U on κ, let V^κ/U denote the ultrapower and $\mathrm{Ult}_U(V)$ its transitive collapse. The canonical embedding will be denoted j_U.

2. The Π^1_n-enforceable filter

Generalized Mahlo operations and their order relation are considered in [7], [8]. For convenience, various facts are reviewed here.

Suppose $n \geq 0$ and κ is a Π^1_n-indescribable cardinal. For $\phi = \phi(\vec{P})$ a Π^1_n sentences with second order parameters such that $\models_{V_\kappa} \phi$, let D_ϕ denote $\{\lambda \in \mathrm{Card} \cap \kappa : \models_{V_\lambda} \phi(P_1 \cap V_\lambda, \ldots, P_k \cap V_\lambda)$. Letting ϕ range over such sentence, the filter generated by the D_ϕ will be called the Π^1_n-enforceable filter, and denoted $\mathcal{E}_n$.

Codes for formulas may be defined in a standard manner, say as hereditarily finite sets; often a code for a formula will simply be called a formula, and denoted ϕ.

Lemma 1. *a. There is a Δ^1_1 formula* $\mathrm{Tru}_0(\phi, A)$ *such that if $\vec{X}$ includes the free variables of ϕ and A is an assignment to $\vec{X}$ then in V_α for any limit ordinal $\alpha > \omega$,* $\mathrm{Tru}_0(\phi, A)$ *iff ϕ is true at A.*

b. For $n \geq 1$ there is a Π^1_n formula $\mathrm{Tru}_n(\phi, A)$ *such that if $\vec{X}$ includes the free variables of ϕ and A is an assignment to $\vec{X}$ then in V_α for any limit ordinal $\alpha > \omega$,* $\mathrm{Tru}_n(\phi, A)$ *iff ϕ is true at A.*

c. There is a Δ^0_1 formula $\mathrm{Sat}(m, \phi, a_2, a_1)$ *such that if m is an $\in$-structure, a_2 (resp. a_1) is an assignment of subsets (resp. elements) of m whose domain includes the free second (resp. first) order variables of ϕ, then in V_α for any limit ordinal $\alpha > \omega$,* $\mathrm{Sat}(m, \phi, a_2, a_1)$ *is true iff ϕ is true in m at the assignment given by a_2 and a_1.*

d. κ is Π^1_0-indescribable iff $\kappa \in \mathrm{Inac}$; and $\mathcal{E}_0$ is the club filter.

e. For $n \geq 1$ $\mathcal{E}_n$ is a proper κ-complete normal filter containing the set of Π^1_{n-1}-indescribable cardinals.

Proof. These are well-known facts, and proofs of most of them can be found in [6]. For ease of reference some details will be given. Formulas may be assumed to be in prenex normal form with distinct bound variables. Δ^0_0 formulas may have bounded quantifiers.

For part a, let $P_1(A)$ be a Δ^1_0 predicate stating that A is an assignment to some second order variables, i.e., its elements are of the form $\langle \ulcorner X_i \urcorner, x\rangle$ where $\ulcorner X_i \urcorner$ is the code for X_i. Let $P_2(T, A)$ be Δ^1_0 predicate stating that T is a "truth assignment" at A, i.e., its elements are of the form $\langle \phi, a, t\rangle$ where a is an assignment to some first order variables including those of ϕ, t is 0 or 1, and the recursion equations for truth are satisfied. Using these and similar predicates, Tru_0 may be written in both Σ^1_1 and Π^1_1 form.

For part b, Tru_n will be written as $\forall X_1 \cdots QX_n M$ where Q is $\forall$ if n is odd and $\exists$ if n is even. Let $P_3(\phi, A, B, \vec{X}$ be a Δ^1_0 formula stating that the assignment B results from A by assigning the variables of the ith block of second order quantifiers of ϕ according to X_i. Let $P_4(\phi, \psi)$ be a Δ^0_1 formula stating that ψ is derived from ϕ by removing second order quantifiers. M may be written using P_3, P_4, and $\mathrm{Tru}_0(\psi, B)$, in Π_1 form if n is odd, else Σ_1 form.

Part c is proved similarly to part a; $P_2(m, t, a_2, a_1)$ is Δ^0_1, and states that t is an assignment to some formulas with first order variable assignment extending a_1 of elements of m, which satisfies the recursion equations for truth.

For part d, expand $L_\in$ with unary predicates $X_1, \ldots, X_k$ and let $\mathring{X}_i \subseteq V_\kappa$ be an interpretation for X_i. By standard arguments, if κ is inaccessible then there is a club of limit ordinals α such that $\langle V_\alpha, \mathring{X}_1 \cap V_\alpha, \ldots, \mathring{X}_k \cap V_\alpha\rangle$ is an elementary substructure of $\langle V_\kappa, \mathring{X}_1, \ldots, \mathring{X}_k\rangle$. In particular D_ϕ for a Π^1_0 sentence ϕ is in the club filter, and κ is Π^1_0-indescribable. If α is not an inaccessible cardinal, then in each of the cases α a successor ordinal, α singular, and α not closed under $\gamma \mapsto 2^\gamma$, a Π^1_0 sentence can be given which holds in V_α but not V_β for $\beta < \alpha$. If C is club let ϕ be the sentence "C is unbounded"; then $\models_{V_\kappa} \phi$, and if $\alpha \in \mathrm{Lim}(C)$ then $\alpha \in D_\phi$. Thus, $C \in \mathcal{E}_0$.

For part e, by definition each $S \in \mathcal{E}_n$ is nonempty so $\mathcal{E}_n$ is proper. By what has been proved, it contains the club filter. It suffices to show that it is closed under diagonal intersection. Say that a set $X \subseteq \kappa$ is enforced by the sentence ϕ if $D_\phi \subseteq X$. Suppose $X_\xi \subseteq \kappa$ for $\xi < \kappa$ is enforced by $\phi_\xi(P_\xi)$. Let ϕ_l be a sentence enforcing "$\alpha > \omega$ is a limit ordinal", let $P = \{\langle \xi, p\rangle : p \in P_\xi\}$, let $Q = \{\langle \xi, \phi_\xi\rangle\}$, and let $T_s(\xi, \phi, A)$ be $\mathrm{Tru}(\phi', A)$ where ϕ' is ϕ with all subformulas $w \in X$ replaced by $\langle \xi, w\rangle \in X$. Then $\triangle_{\xi<\kappa} X_\xi$ is enforced by $\phi_l \wedge \forall\xi\forall\phi(\langle \xi, \phi\rangle \in Q \Rightarrow T_s(\xi, \phi, P))$, which may be transformed to Π^1_n form by well-known methods. Finally, the set of Π^1_{n-1}-indescribable cardinals is enforced by the sentence $\forall\phi\forall A(\mathrm{Tru}(\phi, A) \Rightarrow \exists\alpha \mathrm{Sat}(V_\alpha, \phi, a', \emptyset))$ where if A assigns $\mathring{X}_i$ to X_i then a' assigns $\mathring{X}_i \cap V_\alpha$ to it. □

Let $\mathcal{E}^*_n$ denote the dual ideal; $\mathcal{E}^*_0$ is the thin sets, and $\mathcal{E}^{*c}_0$ the stationary sets. It is easily seen that $X \in \mathcal{E}^{*c}_n$ iff $\forall\phi(\models_{V_\kappa} \phi \Rightarrow D_\phi \cap X \neq \emptyset)$; such X are called Π^1_n indescribable in [7].

Lemma 2. *$\mathcal{E}_n \subseteq \mathcal{E}^{*c}_n$. If κ is Π^1_{n+1}-indescribable then $\mathcal{E}_n \subseteq \mathcal{E}_{n+1}$, $\mathcal{E}^*_n \subseteq \mathcal{E}^*_{n+1}$, and $\mathcal{E}^{*c}_{n+1} \subseteq \mathcal{E}^{*c}_n$. If κ is Δ^1_∞-indescribable then $\mathcal{E}_\infty = \cup_n \mathcal{E}_n \subseteq \cap_n \mathcal{E}^{*c}_n$.*

Proof. Left to the reader. □

Also in [7], for $n \geq 1$ a subset $X \subset \kappa$ for a Π^1_n-indescribable cardinal κ is said to be an n-club iff

1. If $X \cap \alpha \in \mathcal{E}^{*c}_{n-1}$ then $\alpha \in X$ (where $\mathcal{E}_{n-1}$ is taken in α, which is Π^1_{n-1}-indescribable).
2. $X \in \mathcal{E}^{*c}_{n-1}$.

Lemma 3. *Suppose $n \geq 1$ and κ is Π^1_n-indescribable. Then $X \in \mathcal{E}_n$ iff X contains an n-club.*

Proof. This is theorem 2.4.2 of [7]. In one direction, any D_ϕ is an n-club. Indeed, $D_\phi \in \mathcal{E}_n \subseteq \mathcal{E}_{n-1} \subseteq \mathcal{E}^{*c}_{n-1}$. Suppose α is Π_{n-1}-indescribable, $D_\phi \cap \alpha \in \mathcal{E}^{*c}_{n-1}$, and $\models_{V_\alpha} \neg\phi$. Then $D_{\neg\phi} \cap \alpha \in \mathcal{E}_{n-1}$, a contradiction.

In the other direction, suppose X is an n-club. Let ϕ_l be a sentence enforcing "$\alpha > \omega$ is a limit ordinal", Let ϕ_e be a sentence enforcing "α is Π^1_{n-1}-enforceable". Let ϕ be the Π^1_n sentence $\phi_l \wedge \forall\psi\forall A(\mathrm{Tru}_{n-1}(\psi, A) \Rightarrow \exists\alpha \in X \mathrm{Sat}(V_\alpha, \phi_e \wedge \psi, a', \emptyset))$ where if A assigns $\mathring{X}_i$ to X_i then a' assigns $\mathring{X}_i \cap V_\alpha$ to it. Since $X \in \mathcal{E}^{*c}_{n-1}$, $\models_{V_\kappa} \phi$. Suppose $\alpha < \kappa$ and $\models_{V_\alpha} \phi(X \cap V_\alpha)$. Then $X \cap V_\alpha \in \mathcal{E}^{*c}_{n-1}$, and since X is an n-club $\alpha \in X$. This shows $D_\phi \subseteq X$, and so $X \in \mathcal{E}_n$. □

Define a 0-club to be a club. The lemma is true in this case also (lemma 1.d). By what has already been proved, if $\eta < \kappa$ and X_ξ for $\xi < \eta$ is an n-club then $\cap_{\xi<\eta} X_\xi \in \mathcal{E}_n$; in fact, $\cap_{\xi<\eta} X_\xi$ is an n-club. The proof in the case $n = 0$ is well-known; the proof when $n > 0$ is left to the reader. Likewise if X_ξ for $\xi < \kappa$ is an n-club then $\triangle_{\xi<\kappa} X_\xi$ is an n-club.

For κ Π^1_n-indescribable where $n \geq 0$ let $M_n : \mathrm{Pow}(\kappa) \mapsto \mathrm{Pow}(\kappa)$ be the operation where $M_n(X) = \{\alpha < \kappa : X \cap \alpha \in \mathcal{E}^{*c}_n\}$. In particular, if $\alpha \in M^n(X)$ then α is Π^1_n-indescribable. In the case $n = 0$ this differs from a definition in common use as in [9], where only $\mathrm{cf}(\alpha) > \omega$ is required. The requirement $\alpha \in \mathrm{Inac}$ has been used before, in [1] for example.

If κ is the smallest Π^1_n-indescribable cardinal then $M_n(X) = \emptyset$ for all n. On the other hand, the following holds.

Lemma 4. *If κ is Π^1_{n+1}-indescribable and $X \in \mathcal{E}^{*c}_n$ then $M_n(X) \in \mathcal{E}_{n+1}$, whence both $\mathcal{E}_{n+1}$ and $\mathcal{E}^{*c}_n$ are closed under M_n.*

Proof. This is lemma 2.4.3 of [7]. As in the proof of lemma 3 let ϕ be the Π^1_{n+1} sentence $\phi_l \wedge \forall\psi\forall A(\mathrm{Tru}_n(\psi, A) \Rightarrow \exists\alpha \in X \mathrm{Sat}(V_\alpha, \phi_e \wedge \psi, a', \emptyset))$ where ϕ_e enforces "α is Π^1_n-enforceable". Then $\models_{V_\kappa} \phi$, and $D_\phi \subseteq M_n(X)$. □

Define M_{-1} to be Lim; this is defined for any ordinal κ, and $\mathcal{E}_{-1}$ may be taken as $\mathrm{Pow}(\kappa)$.

Lemma 5. *Suppose κ is Π^1_n-indescribable where $n \geq 0$ and $X, Y \subseteq \kappa$.*

a. *If $X \subseteq Y$ then $M_n(X) \subseteq M_n(Y)$ (this holds for M_{-1} also).*
b. *If $\alpha < \kappa$ is Π^1_n-indescribable then $M_n(X \cap \alpha) = M_n(X) \cap \alpha$ (this holds for $n = -1$ also).*
c. *If X is an n-club, $\alpha < \kappa$ is Π^1_n-indescribable, and $\alpha \in M_{n-1}(X)$, then $X \cap \alpha$ is an n-club.*
d. *If X is an n-club then $M_{n-1}(X)$ is an n-club.*
e. *If $M_n(X) \in \mathcal{E}^{*c}_n$ then $X \in \mathcal{E}^{*c}_n$.*
f. *$M_n(M_n(X)) \subseteq M_n(X)$.*
g. *$M_n(X \cup Y) = M_n(X) \cup M_n(Y)$.*
h. *If $X \subseteq \kappa$ and C is an n-club then $M_n(X \cap C) = M_n(X) \cap M_{n-1}(C)$.*

Proof. Parts a to c follow straightforwardly from the definitions. For $n = 0$ part d is well-known (a limit of limit points is a limit point, and unboundedness follows using $\mathrm{cf}(\kappa) > \omega$). For $n > 0$, by lemma 4 $M_{n-1}(X) \in \mathcal{E}^{*c}_{n-1}$; and if $M_{n-1}(X) \cap \alpha \in \mathcal{E}^{*c}_{n-1}$ then by part b $M_{n-1}(X \cap \alpha) \in \mathcal{E}^{*c}_{n-1}$, whence by part d for $n-1$ $X \cap \alpha \in \mathcal{E}^{*c}_{n-1}$, whence since X is an n-club $\alpha \in X$, which completes the proof that $M_{n-1}(X)$ is an n-club. For part e, suppose $M_n(X) \in \mathcal{E}^{*c}_n$. Suppose C is an n-club; then by part d $M_{n-1}(C)$ is an n-club, so there is some $\alpha \in M_{n-1}(C) \cap M_n(X)$. Since $\alpha \in M_{n-1}(C)$, by part c $C \cap \alpha$ is an n-club subset of α; and since $\alpha \in M_n(X)$, $X \cap \alpha \in \mathcal{E}^{*c}_n$. Thus, $C \cap X$ is nonempty, and since C was arbitrary $X \in \mathcal{E}^{*c}_n$. For part f, suppose $\alpha \notin M_n(X)$ where α is Π^1_n-indescribable. Then there is an n-club $C \subseteq \alpha$ such that $C \cap X \cap \alpha = \emptyset$. If β is Π^1_n-indescribable and $\beta \in M_{n-1}(C)$ then $C \cap \beta$ is an n-club, whence $X \cap \beta \in \mathcal{E}^*_n$, whence $\beta \notin M_n(X)$. Since β was arbitrary, $M_{n-1}(C) \cap M_n(X) = \emptyset$, whence $\alpha \notin M_n(M_n(X))$. For part g, by part a $M_n(X) \cup M_n(Y) \subseteq M_n(X \cup Y)$. Suppose $\alpha \in M_n(X \cup Y)$, so that $(X \cup Y) \cap \alpha \in \mathcal{E}^{*c}_n$; then either $X \cap \alpha \in \mathcal{E}^{*c}_n$ or $Y \cap \alpha \in \mathcal{E}^{*c}_n$.

For part h, first note that an element of either side is Π^1_n-indescribable and if $\alpha < \kappa$ is Π^1_n-indescribable, C is an n-club, and $C \cap \alpha \in \mathcal{E}^{*c}_{n-1}$ then $C \cap \alpha$ is an n-club. Then $\alpha \in M_n(X) \cap M_{n-1}(C)$ iff $X \cap \alpha \in \mathcal{E}^{*c}_n$ and $X \cap \alpha \in \mathcal{E}^{*c}_{n-1}$ iff $X \cap C \cap \alpha \in \mathcal{E}^{*c}_n$ iff $\alpha \in M_n(X \cap C)$. □

Lemma 6. *Suppose κ is Π^1_n-indescribable and X_i for $i \in \omega$ is a descending chain under $\subseteq$ of sets in $\mathcal{E}_n$. Then $M_n(\cap_i X_i) = \cap_i M_n(X_i)$.*

Proof. $\alpha \in M_n(\cap_i X_i)$ iff $(\cap_i X_i) \cap \alpha \in \mathcal{E}^{*c}_n$ iff $\cap_i (X_i \cap \alpha) \in \mathcal{E}^{*c}_n$ iff $X_i \cap \alpha \in \mathcal{E}^{*c}_n$ for all i iff $\alpha \in \cap_i M_n(X_i)$. □

If $\mathcal{I}$ is an ideal in a Boolean algebra the relation $X \sqsubseteq_{\mathcal{I}} Y$ is defined to hold iff $X - Y \in \mathcal{I}$. This relation is reflexive and transitive, and its quotient equivalence relation is the usual one $\equiv_{\mathcal{I}}$ on the Boolean algebra, namely $X \equiv_{\mathcal{I}} Y$ iff $X \sqsubseteq_{\mathcal{I}} Y$ and $Y \sqsubseteq_{\mathcal{I}} X$ iff $X - Y \sqcup Y - X \in \mathcal{I}$. Specializing to the Boolean algebra $\mathrm{Pow}(\kappa)$ for a regular uncountable cardinal κ, and writing $\subseteq_{\mathcal{I}}$ for the relation, if $\mathcal{I}$ is κ-complete and $X_\xi \subseteq_{\mathcal{I}} Y_\xi$ for $\xi < \eta$ where $\eta < \kappa$ then $\cap_{\xi<\eta} X_\xi \subseteq_{\mathcal{I}} \cap_{\xi<\eta} Y_\xi$ (this follows because $\cap X_\xi - \cap Y_\xi \subseteq \cup(X_\xi - Y_\xi)$, and if $\mathcal{I}$ is normal and $X_\xi \subseteq_{\mathcal{I}} Y_\xi$ for $\xi < \kappa$ then $\triangle_{\xi<\kappa} X_\xi \subseteq_{\mathcal{I}} \triangle_{\xi<\kappa} Y_\xi$ (this follows because $\triangle X_\xi - \triangle Y_\xi \subseteq \bigtriangledown(X_\xi - Y_\xi)$).

Lemma 7. *Suppose κ is Π^1_n-indescribable. Then if $X \subseteq_{\mathcal{E}^*_n} Y$ then $M_n(X) \subseteq_{\mathcal{E}^*_n} M_n(Y)$.*

Proof. $X \subseteq_{\mathcal{E}^*_n} Y$ then $X \subseteq Y \cup T$ where $T \in \mathcal{E}^*_n$, so $M_n(X) \subseteq M_n(Y \cup T) = M_n(Y) \cup M_n(T)$, so $M_n(X) \subseteq_{\mathcal{E}^*_n} M_n(Y)$. □

Suppose κ is Π^1_n-indescribable. Let $\prec_n$ he the binary relation on $\mathcal{E}^{*c}_n$ where $X \prec_n Y$ iff $Y \subseteq_{\mathcal{E}^*_n} M_n(X)$.

Lemma 8. *Suppose κ is Π^1_n-indescribable.*
a. $\prec_n$ is transitive.
b. $\prec_n$ is well-founded.

Proof. For part a, suppose $Y \subseteq_{\mathcal{E}^*_n} M_n(X)$ and $Z \subseteq_{\mathcal{E}^*_n} M_n(Y)$. Then $M_n(Y) \subseteq_{\mathcal{E}^*_n} M_n(M_n(X))$ and $M_n(M_n(X)) \subseteq M_n(X)$ whence $Z \subseteq_{\mathcal{E}^*_n} M_n(X)$. For part b, suppose $X_0 \succ_n X_1 \succ_n X_2 \succ_n \cdots$ is an infinite descending chain. Let C_i be an n-club disjoint from $X_i - M_n(X_{i+1})$, so that $X_i \cap C_i \subseteq M_n(X_{i+1})$. Let $C'_i = \cap_{j\geq 0} M^j_{n-1}(C_{i+j})$ where $M^j_{n-1}(X)$ denotes M_{n-1}, applied to X j times. Using lemma 5.h repeatedly, $X_i \cap \cap_{j\leq k} M^j_{n-1}(C_{i+j}) \subseteq M_n(X_{i+1} \cap \cap_{j\leq k-1} M^j_{n-1}$ (C_{i+1+j}) for all $k > 0$. Letting $X'_i = X_i \cap C'_i$ and using lemma 6, $X'_i \subseteq M_n(X'_{i+1})$. Let α_i be the least element of X'_i. Then $\alpha_i \in M_n(X'_{i+1})$, so $\alpha_i > \alpha_{i+1}$, which yields an infinite descending chain of ordinals. □

Suppose $m < n$. By lemma 2, $M_n(X) \subseteq M_m(X)$. On the the hand, $X \subseteq_{\mathcal{E}^*_m} Y \Rightarrow X \subseteq_{\mathcal{E}^*_n} Y$. Thus, a comparison of $\prec_m$ and $\prec_n$ is not straightforward.

3. Δ^1_∞ WPS's

As in [3], a binary relation $\preceq$ on some set is said to be a WPS (well-preorder on a subset) if it satisfies the axioms

T1. $X \preceq Y \wedge Y \preceq Z \Rightarrow X \preceq Z$
T2. $X \preceq Y \Rightarrow X \preceq X$
T3. $X \preceq Y \Rightarrow Y \preceq Y$
T4. $X \preceq X \wedge Y \preceq Y \Rightarrow (X \preceq Y \vee Y \preceq X)$
F. For all functions $f : \omega \mapsto \text{Fld}(\preceq)$ there is an n such that $f(n) \preceq f(n+1)$

The following facts are also observed in [3].

- Write $\prec$ for $X \preceq Y \wedge \neg Y \preceq X$; $\prec$ is well-founded. Write $X \equiv Y$ for $X \preceq Y \wedge Y \preceq X$; $\equiv$ is a congruence relation for $\preceq$ and $\prec$. The quotient $\prec / \equiv$ is a well-order in its field.
- Write $\Omega(X)$ for the rank of $X \in \mathrm{Fld}(\prec)$, and $\Omega(\prec)$ for the rank of $\prec$. $\Omega(\prec) = \Omega(\prec / \equiv)$, and the latter is the order type.
- Write $X \preceq_A Y$ for $X \prec A \wedge Y \prec A \wedge X \preceq Y$; $\preceq_A$ is defined even if $A \notin \mathrm{Fld}(\preceq)$, and equals $\emptyset$. $\preceq_A$ is a WPS, and $\Omega(\prec_A) = \Omega(A)$.

A WPS on V_κ is said to be Δ^1_∞ if it is defined in V_κ by a Δ^1_∞ formula (in two free variables).

Given $\kappa \in \mathrm{Card}$, let $B_{\Delta^1_\infty}$ denote $\sup\{\Omega(\prec) : \preceq \text{ is a } \Delta^1_\infty \text{ WPS}\}$. It is readily seen that $B_{\Delta^1_\infty}$ is a limit ordinal and if $\alpha < B_{\Delta^1_\infty}$ then $\alpha = \Omega(\preceq)$ for some Δ^1_∞ WPS $\preceq$. Thus, $\alpha < B_{\Delta^1_\infty}$ iff for some n and Π^1_n WPS $\preceq$, $\alpha = \Omega(\preceq)$.

4. Set Chains

The statement that a Π^1_n formula in two free second order variables defines a WPS is a Π^1_{n+1} sentence ϕ. For a Π^1_{n+1}-indescribable cardinal κ and a Π^1_n WPS in V_κ let $D_\preceq$ denote D_ϕ.

Theorem 9. *Suppose $\preceq$ is a Π^1_n WPS with $\alpha = \Omega(\prec)$, and κ is Π^1_{n+1}-indescribable. Then there is a sequence S_ξ for $\xi < \alpha$ of elements of $\mathcal{E}_{n+1}$ which is a chain under $\prec_n$.*

Proof. For this proof, let M'_n be M_n as defined earlier, and let $M_n(X) = X \cap M'_n(X)$. Define $\prec_n$ using the new operator; it suffices to construct a chain under this relation.

The definition of $\mathrm{H}^A(X)$ from section 7 of [5] may be generalized. Suppose κ is Π^1_n-indescribable, $S \subseteq \kappa$ is a set of Π^1_n-indescribables, $\prec$ is a Σ^1_{n+1}-WF reflecting in S, $A \in \mathrm{Fld}(\prec)$, and $X \subseteq S$. Say that $\lambda \in M^A_n(X)$ iff $\lambda \in X$ and $M^p_n(X \cap \lambda)$ is a $\mathcal{E}^{*c}_n$ subset of λ for all $p \in \mathrm{Fld}(\prec_\lambda)$ where $p \prec_\lambda A \cap V_\lambda$.

Lemma 16 and theorem 17 of [5] generalize. If $A \prec B$ then this reflects in $C \in \mathcal{E}_n$. For any $X \subseteq S \cap C$, $M^B_n(X) \subseteq_{\mathcal{E}^*_n} M_n(M^A_n(X))$.

Lemma 18 of [5] generalizes. There is a Δ^1_0 formula $\phi_M(\phi, \lambda, A, B, S, P)$ such that if $\phi(A, B, P)$ is a Σ_{n+1} formula defining a WF $\prec$ which reflects in a set $S \subseteq \kappa$ of Π^1_1-indescribables, $\lambda \in S$, $A \in \mathrm{Fld}(\prec)$, and $X \subseteq S$, then $\models_{V_\kappa} \phi_M$ iff $\lambda \in M^A_n(X)$.

Since κ is Π^1_{n+1}-indescribable, $\prec$ is a Σ^1_{n+1}-WF which reflects in a set $S \in \mathcal{E}_{n+1}$ of Π^1_n-indescribables (namely $D_\preceq$).

It remains to show that for all A $M^A_n(S) \in \mathcal{E}_{n+1}$. This follows by generalizing facts noted in theorem 19 of [5], by noting that $\phi_\prec$ is Π^1_{n+1}. □

The theorem can be strengthened; there is an m such that for $n \geq m$ there is a chain in $\prec_n$.

Suppose κ is a measurable cardinal. It is well-known ([6]) that κ is Δ^1_∞-indescribable. Further if U is a normal ultrafilter on κ then $\cup_n \mathcal{E}_n \subseteq U \subseteq \cap_n \mathcal{E}^{*c}_n$. (whence for a Δ^1_∞ WPS $\preceq$, $D_\preceq \in U$ for any ultrafilter U on κ). In particular, if in the theorem κ is measurable then $S_x i$ is a chain in any normal ultrafilter U.

Theorem 10. *Suppose $\preceq$ is a Δ^1_∞ WPS on V_κ. and U is an ultrafilter on κ.*

a. The function $\lambda \mapsto \preceq_\lambda$ on $D_\preceq$ represents $\preceq$ in $\mathrm{Ult}_U(V)$.

b. The function $\lambda \mapsto \Omega(\preceq_\lambda)$ on $D_\preceq$ represents $\Omega(\preceq)$.

Proof. Let ϕ be the formula defining $\preceq$. Given X, Y, let $D_+ = \{\lambda \in \mathrm{Card} \cap \kappa : \models_{V_\lambda} \phi(X \cap V_\lambda, Y \cap V_\lambda)\}$ and $D_- = \{\lambda \in \mathrm{Card} \cap \kappa : \models_{V_\lambda} \neg\phi(X \cap V_\lambda, Y \cap V_\lambda)\}$. Then $X \preceq Y \Rightarrow D_+ \in \mathcal{E}_\infty \Rightarrow D_+ \in U$, and $\neg X \preceq Y \Rightarrow D_- \in \mathcal{E}_\infty \Rightarrow D_- \in U \Rightarrow D_+ \notin U$. This proves part a. For part b, the function represents $\Omega(\preceq)$ in $\mathrm{Ult}_U(V)$, and part b follows by Δ_1 absoluteness. □

The Mitchell order on normal ultrafilters on a measurable cardinal κ is defined in [11]. For normal ultrafilters U_1, U_2, $U_1 \triangleleft U_2$ iff $U_1 \in \mathrm{Ult}_{U_2}(V)$. This relation is transitive and well-founded [11]. The rank of a normal ultrafilter U in this relation is denoted $o(U)$, and the rank of $\triangleleft$ is denoted $o(\kappa)$; $o(\kappa)$ may be defined to be 0 for a cardinal κ which is not measurable.

By a sequence of normal ultrafilters on κ will be meant a function $\mathcal{U}$ whose domain is some ordinal η, such that for $\beta < \eta$, $\mathcal{U}(\beta)$ is a normal ultrafilter on κ with $o(\mathcal{U}(\beta)) = \beta$.

Theorem 11. *Suppose $\preceq$ is a Π^1_n WPS with $\alpha = \Omega(\prec)$, $o(\kappa) \geq \alpha$, and $\mathcal{U}$ is a sequence of normal ultrafilters on κ with domain α. Then there is a sequence S_ξ for $\xi < \alpha$ where $S_\xi \in \mathcal{U}(\xi)$ which is a chain under $\prec_n$.*

Proof. Given $A \in \mathrm{Fld}(\preceq)$ let f_A be the function $\lambda \mapsto \Omega(\preceq_{A,\lambda})$ on $D_{\preceq}$ as in theorem 10. Given $\xi < \alpha$ choose A with $\Omega(\prec_A) = \xi$, and let f_ξ be f_A, so that f_ξ represents ξ in any normal ultrafilter on κ.

The theorem may be proved by adapting the proof of theorem 26 of [2]. For $\xi < \alpha$ let S_ξ denote $\{\lambda \in D_{\preceq} : o(\lambda) \geq f_\xi(\lambda)\}$. o and f_ξ represent ξ mod $\mathcal{U}(\xi)$, and $S_\xi \in \mathcal{U}(\xi) \subseteq \mathcal{E}_n^{*c}$ follows.

For $\lambda \in D_{\preceq}$, it is readily seen that $f_\xi \restriction \lambda$ equals the function obtained in V_λ from $\preceq_{A,\lambda}$, whence $f_\xi \restriction \lambda$ represents $f_\xi(\lambda)$ mod any normal ultrafilter on λ.

Suppose $\lambda \in S_{\xi+1}$ (where $\xi = \Omega(\prec_A)$ and $\xi + 1 < \alpha$). Then $o(\lambda) \geq f_{\xi+1}(\lambda)$, so there is a normal ultrafilter U' on λ with $o(U') = f_\xi(\lambda)$. If $\lambda \in S_\xi$ also then $\{\mu < \lambda : o(\mu) = f_{A \cap V_\lambda}(\mu)\} \in U'$, whence $S_\xi \cap \lambda \in \mathcal{E}_n^{*c}$, whence $\lambda \in M_n(S_\xi)$. The theorem follows from this fact and the fact that $\xi < \zeta \Rightarrow S_\zeta \subseteq_{\mathcal{E}_n^*} S_\xi$. □

Note that theorems 10 and 11 improve on theorem 25 and 26 of [2] in the case $n = 0$.

5. Repeat points

In this section κ will denote a measurable cardinal. Repeat points for coherent sequences of normal ultrafilters were defined by Mitchell [12]. Subsequently they have appeared in various references, for example [10], [15], [14]. Here, repeat points will be defined for a sequence $\mathcal{U}$ of normal ultrafilters on κ.

Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ. An ordinal $\beta < \mathrm{Dom}(\mathcal{U})$ is said to be a repeat point of $\mathcal{U}$ if $\mathcal{U}(\beta) \subseteq \bigcup_{\gamma<\beta} \mathcal{U}(\gamma)$. Theorem 11 gives a lower bound on the smallest repeat point. A more direct bound is of interest, and theorem 11 can be seen as a refinement.

For an ultrafilter U on κ and $f : \kappa \mapsto V$, as usual let $[f]$ denote indifferently the element of V^κ/U or $\mathrm{Ult}_U(V)$. The notation $[f]_U$ may be used to indicate U.

Lemma 12. *Suppose U is a normal ultrafilter on κ and let $D_f^= = \{\lambda \in \mathrm{Card} \cap \kappa : f(\lambda) = o(\lambda)\}$. Then $o(U) = [f]$ iff $[o] = [f]$ iff $D_f^= \in U$.*

Proof. It is well known ([13]) that $[o] = o(U)$; the lemma follows readily. $\square$

Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ, and $I \subseteq [0, \mathrm{Dom}(\mathcal{U}))$ is an interval. Say that a function $f \in V^\kappa$ representing f on I if $[f]_{\mathcal{U}(\beta)} = \alpha$ for $\beta \in I$.

Theorem 13. *Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ, $\alpha < Dom(\mathcal{U})$, and f represents α on $[0, \alpha]$. Then α is not a repeat point of $\mathcal{U}$.*

Proof. By lemma 12 $D^{=}_{f} \in \mathcal{U}(\alpha)$. Suppose $D^{=}_{f} \in \mathcal{U}(\beta)$ where $\beta < \alpha$. Again by lemma 12 $[f]_{\mathcal{U}(\beta)} = \beta$, contradicting the hypothesis. $\square$

Corollary 14. *For $\alpha < Dom(\mathcal{U})$, if $\alpha < B_{\Delta^1_\infty}$ then α is not a repeat point of $\mathcal{U}$.*

Proof. This follows from the theorem by theorem 10.b. $\square$

Corollary 15. *If for any $\mathcal{U}$ and $\alpha \in Dom(\mathcal{U})$ there is a function representing α on $[0, \alpha]$ then repeat points do not exist.*

Proof. Immediate. $\square$

The author is not aware of an explicit previous statement of theorem 13, but the method is well-known. Given $\mathcal{U}$ and $\alpha < \mathrm{Dom}(\mathcal{U})$, say that $f \in V^\kappa$ E-represents α if $[f]_{\mathcal{U}}(\beta)$ equals β if $\beta < \alpha$, else α. For $f \in V^\kappa$ let $D^{<}_{f} = \{\lambda : f(\lambda) < o(\lambda)$.

Theorem 16. *Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ, $\alpha < \nu < Dom(\mathcal{U})$, f_1 E-represents α, and f_2 represents ν on $[\alpha, \nu]$. Then ν is not a repeat point of $\mathcal{U}$.*

Proof. The proof is as in the proof of lemma 3.8 of [15]. For any f and U, $o(U) < [f]$ iff $[o] < [f]$ iff $D^{<}_{f} \in U$. If f E-represents α, it follows that $D^{<}_{f} \in \mathcal{U}(\beta)$ iff $\beta > \alpha$. It then follows that $D^{<}_{f_1} \cap D^{=}_{f_2}$ is in $\mathcal{U}(\nu)$ but not $\mathcal{U}(\beta)$ for $\beta < \nu$. $\square$

Theorem 17. *Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ, $\alpha < Dom(\mathcal{U})$, f_1 E-represents α, and f_2 represents α on $[0, \alpha]$. Then there is an f representing α on $[0, Dom(\mathcal{U}))$.*

Proof. Let $f(\lambda)$ equal $f_1(\lambda)$ if $f_1(\lambda) < o(\lambda)$, else $f_2(\lambda)$. Then $\alpha < \beta \Rightarrow D^<_{f_1} \in \mathcal{U}(\beta) \Rightarrow [f]_{\mathcal{U}(\beta)} = [f_1]_{\mathcal{U}(\beta)} = \alpha$; and $\alpha \geq \beta \Rightarrow D^<_{f_1} \notin \mathcal{U}(\beta) \Rightarrow [f]_{\mathcal{U}(\beta)} = [f_2]_{\mathcal{U}(\beta)} = \alpha$. □

Theorem 18. *Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ, and α is any ordinal. Then there is an f representing α on $[0, \min(\kappa, Dom(\mathcal{U})))$.*

Proof. For $\beta \in [0, \min(\kappa, \mathrm{Dom}(\mathcal{U})))$ let $D_\beta = \{\lambda \in \mathrm{Inac} \cap \kappa : o(\lambda) = \beta\}$ and let f_β be any function in V^κ such that $[f_\beta]_{\mathcal{U}(\beta)} = \alpha$. Let f be the function where $f(\lambda) = f_{o(\lambda)}(\lambda)$. Since $\beta < \kappa$, $D_\beta \in U$ iff $o(U) = \beta$ (lemma 12). Thus $[f]_{\mathcal{U}(\beta)} = [f_\beta]_{\mathcal{U}(\beta)} = \alpha$. □

Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ. Let $B_{RFd}(\mathcal{U})$ (resp. $B_{RF}(\mathcal{U})$) be the smallest α such that α does not have a function representing it on $[0, \alpha]$ (resp. $[0, \mathrm{Dom}(\mathcal{U}))$) (or $\mathrm{Dom}(\mathcal{U})$ if there is no such α). Let $B_{RP}(\mathcal{U})$ be the smallest repeat point (or $\mathrm{Dom}(\mathcal{U})$ if there is no such). By results above, $B_{\Delta^1_\infty} \leq B_{RF}(\mathcal{U}) \leq B_{RFd}(\mathcal{U}) \leq B_{RP}(\mathcal{U})$.

Theorem 19. *Suppose $\mathcal{U}$ is a sequence of normal ultrafilters on κ. If $\mathrm{cf}(Dom(\mathcal{U}) = \kappa^+$ then $\mathrm{cf}(B_{RF}(\mathcal{U}) = \kappa^+$.*

Proof. If f represents α on $[0, \mathrm{Dom}(\mathcal{U}))$ and $\alpha + 1 < \mathrm{Dom}(\mathcal{U}(\kappa))$ then $f + 1$ represents $\alpha + 1$ on $[0, \mathrm{Dom}(\mathcal{U}))$. If f_ξ represents α_ξ on $[0, \mathrm{Dom}(\mathcal{U}))$ for $\xi < \eta$ where $\eta < \kappa$ and $\alpha = sup_{\xi<\eta}\alpha_\xi < \mathrm{Dom}(\mathcal{U})$ then $\sup_{\xi<\eta} f_\xi$ represents α on $[0, \mathrm{Dom}(\mathcal{U}))$. If f_ξ represents α_ξ on $[0, \mathrm{Dom}(\mathcal{U}))$ for $\xi < \kappa$ and $\alpha = sup_{\xi<\kappa}\alpha_\xi < \mathrm{Dom}(\mathcal{U})$ then $\sup^d_{\xi<\kappa} f_\xi$ represents α on $[0, \mathrm{Dom}(\mathcal{U}))$, where $\sup^d$ is the "diagonal supremum" $f(\lambda) = \sup_{\xi<\lambda} f_\xi(\lambda)$. □

6. Coherent sequences

By a sequence of normal ultrafilters is meant a function $\mathcal{U}$ whose domain is a set (or class) of measurable cardinals, such that for $\kappa \in \mathrm{Dom}(\mathcal{U})$, $\mathcal{U}(\kappa)$ is a sequence of normal ultrafilters on κ.

The notion of a coherent sequence of normal ultrafilters was defined in [11]. A more general definition, given later by various authors, is as follows. A sequence $\mathcal{U}$ is coherent if for all $\kappa \in$

Dom($\mathcal{U}$), for all $\alpha <$ Dom($\mathcal{U}(\kappa)$), $j_{\mathcal{U}(\kappa)(\alpha)}(\mathcal{U}) \restriction \kappa = \mathcal{U} \restriction \kappa$ and $j_{\mathcal{U}(\kappa)(\alpha)}(\mathcal{U})(\kappa) \restriction \alpha = \mathcal{U}(\kappa) \restriction \alpha$.

In [15] some results on repeat points in coherent sequences are given; these will be reviewed here. First, a preliminary fact will be noted.

Lemma 20. *Suppose κ is a measurable cardinal.*

a. *If $U \lhd W$ then $j_U(\kappa) < j_W(\kappa)$.*

b. *If $\forall\gamma < \beta(U_\gamma \lhd W)$ where $\gamma \mapsto o(U_\gamma)$ is increasing and $o(W) = \sup_{\gamma<\beta} o(U_\gamma)$, then $j_W(\kappa) = \sup_{\gamma<\beta} j_{U_\gamma}(\kappa)$.*

Proof. Part a is well-known; see [11] for example. For a proof, suppose U is represented mod W by $\lambda \mapsto U_\lambda$. Then $j_U(\kappa)$ is represented by $\lambda \mapsto j_{U_\lambda}(\lambda)$ and $j_{U_\lambda}(\lambda) < \kappa$. For part b, using similar notation, suppose $\forall\gamma < \beta(j_{U_\gamma}(\kappa) < \alpha)$. Then for some $X \in W$, for all $\lambda \in X$, $\forall\gamma < \beta_\lambda(U_{\gamma,\lambda} \lhd W_\gamma \wedge j_{U_{\gamma,\lambda}}(\lambda) < \alpha_\lambda)$. For each $\lambda \in X$, inductively on κ, $j_{W_\lambda}(\lambda) \leq \alpha_\lambda$. Thus, $j_W(\kappa) \leq \alpha$. □

Thus, if $\mathcal{U}$ is a coherent sequence, for any $\kappa \in$ Dom($\mathcal{U}$), the map $\alpha \mapsto j_{\mathcal{U}(\kappa)(\alpha)}(\kappa)$ is increasing and continuous. This map will be denoted as C, and C^f will be used to denote the fixed point enumerator.

Theorem 21. *[15] Suppose $\mathcal{U}$ is a coherent sequence and $\kappa \in$ Dom($\mathcal{U}$).*

a. *If α is not a repeat point of $\mathcal{U}(\kappa)$ then there is an E-representing function for α.*

b. *If α has an E-representing function and $\nu < C(\alpha)$ then there is a function representing ν on $[\alpha, Dom(\mathcal{U}(\kappa)))$.*

Proof. Part a is proved in lemma 3.7 of [15], and part b in proposition 3.9. □

Theorem 22. *Suppose $\mathcal{U}$ is a coherent sequence and $\kappa \in$ Dom($\mathcal{U}$). If there is a repeat point let θ be the smallest such.*

a. *$B_{RF}(\mathcal{U}(\kappa)) = B_{RFd}(\mathcal{U}(\kappa))$.*

b. *If $\alpha < \theta$ then $\alpha + 1 < \theta$.*

c. *$C(\theta) = \theta$.*

d. *Letting S denote the sequences of ordinals, suppose $f : S \mapsto$ Ord is Σ_1^{ZFC}. If Dom(s), $s_\xi < \theta$ then $f(s) < \theta$.*

e. *$\mathrm{cf}(\theta) = \kappa^+$.*

f. $C^f(\kappa^+) \leq \theta$.
g. $B_{RF}(\mathcal{U}(\kappa)) \geq \min(Dom(\mathcal{U}(\kappa)), C(\kappa))$.

Proof. Part a follows by theorems 13, 21.a, and 17. For part b, by theorem 21.a α has an E-representing function f_1. Let $f_2 = f_1 + 1$; then f_2 represents $\alpha + 1$ on $[\alpha, \mathrm{Dom}(\mathcal{U}(\kappa)))$. The claim follows by theorem 16. For part c, by theorems 21 and 16, if $\alpha < \theta$ then $[\alpha, C(\alpha)) \subseteq \theta$. Further, by part b $\alpha+1 < \theta$, and since $C(\alpha) < C(\alpha+1)$, $C(\alpha) < \theta$. Since C is increasing and continuous there is a unique α such that $\theta \in [C(\alpha), C(\alpha+1))$. Suppose $C(\alpha) < \theta$. Then $\alpha < \theta$ so $\alpha+1 < \theta$ so $C(\alpha+1) < \theta$, a contradiction. Thus $\theta = C(\alpha)$. If $\alpha < \theta$ then $C(\alpha) < \theta$, again a contradiction. Thus $\alpha = \theta$ and $\theta = C(\theta)$. Part d is proved in the remark following proposition 3.9 in [15]. Part e follows similarly to theorem 19, noting that sup and diagonal supremum are Σ_1^{ZFC} and using part d. Part f follows because $\mathrm{cf}(C^f(\alpha))$ equals ω if α is 0 or a successor ordinal, else $\mathrm{cf}(\alpha)$. For part g, if $\alpha < \kappa$ and $\nu < C(\alpha)$ then α is not a repeat point whence by theorem 21 there is a function representing ν on $[\alpha, \mathrm{Dom}(\mathcal{U}(\kappa)))$. This can be used to construct a function representing ν on $[0, \mathrm{Dom}(\mathcal{U}(\kappa)))$ as in the proof of theorem 18. The claim follows by continuity. □

7. Mitchell's model

In [11] Mitchell constructed an inner model where the normal ultrafilters comprise a coherent sequence. In [13] the construction was improved.

If $\mathcal{U}$ is a sequence of normal ultrafilters let $L[\mathcal{U}]$ denote $L[\{\langle\kappa, \alpha, x\rangle : x \in \mathcal{U}(\kappa)(\alpha)\}]$. Let $\overline{\mathcal{U}}(\kappa)(\alpha) = \mathcal{U}(\kappa)(\alpha) \cap L[\mathcal{U}]$; then $L[\mathcal{U}] = L[\overline{\mathcal{U}}]$.

Say that $\mathcal{U}$ is maximal if $\mathrm{Dom}(\mathcal{U})$ is all measurable cardinals, and $\mathrm{Dom}(\mathcal{U}(\kappa)) = o(\mathcal{U}(\kappa))$ for all κ. A sequence $\mathcal{W}$ is said to be a reduction of $\mathcal{U}$ if $\mathrm{Dom}(\mathcal{W}) = \mathrm{Dom}(\mathcal{U})$ and $\mathrm{Dom}(\mathcal{W}(\kappa)) \subseteq \mathrm{Dom}(\mathcal{U}(\kappa))$ for each κ.

Theorem 23. *Suppose $\mathcal{U}$ is a maximal sequence of normal ultrafilters. There is a reduction $\mathcal{U}_r$ of $\mathcal{U}$, such that in $L[\mathcal{U}_r]$ $\overline{\mathcal{U}}_r$ is a coherent sequence containing every normal ultrafilter. Further, neither $\kappa \mapsto Dom(\mathcal{U}(\kappa))$, $L[\mathcal{U}_r]$, nor $\overline{\mathcal{U}}_r$ depends on $\mathcal{U}$, and for each κ $Dom(\mathcal{U}_r(\kappa)) = \min(Dom(\mathcal{U}(\kappa)), (\kappa^{++})^{L[\mathcal{U}_r]})$.*

Proof. The claims may all be verified from the proof of theorem 1 of [13]. □

Theorem 24. *If $\overline{\mathcal{U}}_r$ contains no repeat points then $\mathcal{U}$ contains no repeat points.*

Proof. If $\overline{\mathcal{U}}_r$ contains no repeat points then for each κ $\mathrm{Dom}(\mathcal{U}_r(\kappa)) = \mathrm{Dom}(\mathcal{U}(\kappa))$. If α is a repeat point of $\mathcal{U}(\kappa)$ then $\overline{\mathcal{U}}_r(\kappa)(\alpha) = \mathcal{U}(\kappa)(\alpha) \cap L[\mathcal{U}_r] \subseteq (\cup_{\beta<\alpha}\mathcal{U}(\kappa)(\beta)) \cap L[\mathcal{U}_r] = \cup_{\beta<\alpha}(\mathcal{U}(\kappa)(\beta) \cap L[\mathcal{U}_r]) = \cup_{\beta<\alpha}\overline{\mathcal{U}}_r(\kappa)(\beta)$. □

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